Ordinals in Scunak

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What is Scunak?

**Scunak is**

- Like Automath, Twelf and Coq: Dependent Type Theory, Proof Terms
- Like Coq, Isabelle, and TPS: Interactive Proving Facilities
- Like Mizar: Mathematics encoded in Set Theory

**but**

- not Automath, Twelf, Coq, TPS, Isabelle: Different Type Theory
- not TPS, Isabelle-HOL: Set Theory, not HOL
- not Isabelle-ZF, Mizar: Different Set Theory
- not Mizar: Computes Proof Objects.
Scunak Type Theory

Terms: $x|c|((\lambda xM)|(MN)|\langle M,N\rangle | \pi_1(M) | \pi_2(M)$

Types: $\text{obj} | \text{prop} | \vdash \phi \mid \{x|\phi(x)\} \mid \Pi x : A.B$

Like LF except:

- Restricted to three type families: $\text{obj}$ (all mathematical objects); $\text{prop}$ (all propositions); $\vdash \phi$ (proofs of $\phi$)
- Proof irrelevance: at most one inhabitant of $\vdash \phi$
- Special sum types, the class type $\{x|\phi(x)\}$, equivalent to $\Sigma x:\text{obj} \vdash \phi(x)$
- Restricted to second-order $\lambda$-calculus.
Set Theory in Scunak

**Axiomatic Basis:** 30 basic concepts (propositional constructors, object constructors, proof constructors)

Mac Lane Set Theory, with (global) choice, plus universes, but without foundation.
Set Theory in Scunak

**Axiomatic Basis:** 30 basic concepts (propositional constructors, object constructors, proof constructors)

3 basic propositional constructors:

- **Negation**
  \[ \neg \phi \text{ where } \phi : \text{prop} \]

- **Equality between objects (i.e., sets)**
  \[ x = y \text{ where } x, y : \text{obj} \]

- **Membership**
  \[ x \in y \text{ where } x, y : \text{obj} \]
Set Theory in Scunak

Axiomatic Basis: 30 basic concepts (propositional constructors, object constructors, proof constructors)

7 constructors for objects (sets):

- The empty set $\emptyset$
- Separation $\{ x \in A | \phi(x) \}$
- Set adjunction $(x; A)$, intuitively, $\{ x \} \cup A$
- Powerset $\mathcal{P}(A)$
- Set Union $\bigcup A$
- Universes $Univ(A)$
  A universe is a set containing $A$ closed under the operations above.
- (Global) Choice...
Proof Rules

Remaining 20 basic concepts are natural deduction rules.
Proof Rules

Remaining 20 basic concepts are natural deduction rules.

For Example, Set Adjoin Elimination:

\[
\begin{array}{c}
\text{If } x \in (\{A\} \cup B) \text{ and } \phi \text{ is provable both under the assumption } x = A \text{ and under the assumption } x \in B, \text{ then } \phi \text{ is provable.}
\end{array}
\]
Purity of Set Theory Axioms

All the basic concepts, including rules, are given without using any abbreviations.

1. All 30 constants are declared.
2. Everything after that is an abbreviation.
Some Defined Concepts

Logical Operators: $\land, \lor, \supset$

Bounded, Dependent, Quantifiers: $\forall x \in A. \phi(x)$ and $\exists x \in A. \phi(x)$

Set Theory Relations: $\subseteq$

Set Theory Operations: $\cup, \cap$
Natural Numbers where \(2\) behaves like \(<\).

Like \(0\), i.e., \(0 < 1\), like \(1\), i.e., \(1 < 2\), Successors are formed by adjoining the set to itself:

\[ A \cup A, \text{ i.e., } (A \cup A) \]

Limit Ordinals (e.g., \(\omega\)) equal \(S_X\) where \(X\) is an unbounded set of ordinals.

\[ \emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \ldots, \omega, \{\omega\} \cup \omega, \ldots \]
Ordinals

\[ \emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \ldots, \omega, \{\omega\} \cup \omega, \ldots \]

\( \omega \): Natural Numbers where \( \in \) behaves like \( < \).
Ordinals

\[ \emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \ldots, \omega, \{\omega\} \cup \omega, \ldots \]

\( \omega \): Natural Numbers where \( \in \) behaves like \( < \).

\( \emptyset \): Like 0
Ordinals

\[ \emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \ldots, \omega, \{\omega\} \cup \omega, \ldots \]

\( \omega \): Natural Numbers where \( \in \) behaves like \( < \).

\( \emptyset \): Like 0

\( \{\emptyset\} \), i.e., \{0\}, like 1
Ordinals

$\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}, \emptyset\}$, $\ldots$, $\omega$, $\{\omega\} \cup \omega$, $\ldots$

$\omega$: Natural Numbers where $\in$ behaves like $\prec$.

$\emptyset$: Like 0

$\{\emptyset\}$, i.e., $\{0\}$, like 1

$\{\{\emptyset\}, \emptyset\}$, i.e., $\{1, 0\}$, like 2
Ordinals

\[ \emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \ldots, \omega, \{\omega\} \cup \omega, \ldots \]

\(\omega\): Natural Numbers where \(\in\) behaves like \(<\).

\(\emptyset\): Like 0

\(\{\emptyset\}\), i.e., \(\{0\}\), like 1

\(\{\{\emptyset\}, \emptyset\}\), i.e., \(\{1, 0\}\), like 2

Successors are formed by *adjoining* the set to itself:

\(\{A\} \cup A\), i.e., \((A; A)\).
Ordinals

\[ \emptyset, \{ \emptyset \}, \{ \{ \emptyset \}, \emptyset \}, \ldots, \omega, \{ \omega \} \cup \omega, \ldots \]

\( \omega \): Natural Numbers where \( \in \) behaves like \( < \).

\( \emptyset \): Like 0

\( \{ \emptyset \}, \text{i.e., } \{ 0 \}, \text{like } 1 \)

\( \{ \{ \emptyset \}, \emptyset \}, \text{i.e., } \{ 1, 0 \}, \text{like } 2 \)

Successors are formed by *adjoining* the set to itself:

\( \{ A \} \cup A, \text{i.e., } (A; A). \)

Limit Ordinals (e.g., \( \omega \)) equal \( \bigcup X \) where \( X \) is an unbounded set of ordinals.
What is an Ordinal?

**Defn:** A set \( \alpha \) is an *Ordinal* if

1. \( \alpha \) is “transitive” (it isn’t *that* transitive)
   - \( y \in \alpha \) implies \( y \subseteq \alpha \)

2. \( \alpha \) is well-ordered by \( \epsilon \), i.e.:
   - \( \alpha \) is strictly totally ordered by \( \epsilon \)
   - Every nonempty subset \( Y \) of \( \alpha \) has a \( \epsilon \)-least element.
What is an Ordinal?

Defn: A set $\alpha$ is an *Ordinal* if

- $\alpha$ is “transitive” (it isn’t *that* transitive)
  
  $y \in \alpha$ implies $y \subseteq \alpha$

- $\alpha$ is well-ordered by $\in$, i.e.:
  
  $\alpha$ is strictly totally ordered by $\in$

  Every nonempty subset $Y$ of $\alpha$ has a $\in$-least element.

Ill-typed in HOL.
What is an Ordinal?

**Defn:** A set $\alpha$ is an *Ordinal* if

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- $\alpha$ is well-ordered by $\in$, i.e.:
  
  $\alpha$ is strictly totally ordered by $\in$
  
  Every nonempty subset $Y$ of $\alpha$ has a $\in$-least element.

**Scunak Abbreviations:**

```
[alpha:obj]
(transitiviset alpha):prop
=(forall y:alpha .  (y<=alpha)).
```
What is an Ordinal?

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  - $y \in \alpha$ implies $y \subseteq \alpha$

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  - $\alpha$ is strictly totally ordered by $\in$
  - Every nonempty subset $Y$ of $\alpha$ has a $\in$-least element.

**Scunak Abbreviations:**

$\forall \alpha : \text{obj}
\begin{align*}
\forall \alpha : \text{obj} \quad & \forall \alpha : \text{obj} : \text{prop} \\
& = (\forall y : \alpha . (y \subseteq \alpha)).
\end{align*}$

**Parsed and Reconstructed into:**

$\forall \alpha : \text{obj} : \text{prop}
\begin{align*}
& = \lambda \alpha (\forall \alpha (\forall y (\text{subset} \pi_1 (y) \alpha)))
\end{align*}$
What is an Ordinal?

**Defn:** A set $\alpha$ is an *Ordinal* if

- $\alpha$ is “transitive” (it isn’t *that* transitive)
  
y $\in \alpha$ implies $y \subseteq \alpha$

- $\alpha$ is well-ordered by $\in$, i.e.:
  
  $\alpha$ is strictly totally ordered by $\in$
  
  Every nonempty subset $Y$ of $\alpha$ has a $\in$-least element.

**Scunak Abbreviations:**

$$(\text{stricttotalorderedByIn alpha}) : \text{prop}$$

$$= ((\forall a \ b \ c : \alpha .$$

  $$((\forall a \ b \ c : \alpha . (a::b) \land (b::c)) \implies (a::c))) \land (\forall a \ b : \alpha . ((a\approx b) \lor ((a::b) \lor (b::a)))) \land (\forall a : \alpha . (\not (a::a))))).$$
What is an Ordinal?

**Defn:** A set $\alpha$ is an *Ordinal* if

- $\alpha$ is "transitive" (it isn’t *that* transitive)
  
  $y \in \alpha$ implies $y \subseteq \alpha$

- $\alpha$ is well-ordered by $\epsilon$, i.e.:
  
  $\alpha$ is strictly totally ordered by $\epsilon$
  
  Every nonempty subset $Y$ of $\alpha$ has a $\epsilon$-least element.

Scunak Abbreviations:

$\text{wellorderedByInIn}(\alpha) : \text{prop}$

$= (\text{stricttotalorderedByInIn}(\alpha) \land (\forall Y : (\text{powerset}(\alpha)) .

\quad ((\text{nonempty} \ Y) \Rightarrow
\quad (\exists a : Y . (\forall b : Y . ((a = b) \lor (a : b)))))))).$
What is an Ordinal?

Defn: A set $\alpha$ is an *Ordinal* if

- $\alpha$ is “transitive” (it isn’t *that* transitive)
  \[ y \in \alpha \text{ implies } y \subseteq \alpha \]
- $\alpha$ is well-ordered by $\in$, i.e.:
  \[ \alpha \text{ is strictly totally ordered by } \in \]
  Every nonempty subset $Y$ of $\alpha$ has a $\in$-least element.

Scunak Abbreviations:

(ordinal alpha):prop
\[= ((transitiveset alpha) \&
\quad (wellorderedByIn alpha)).\]
What is an Ordinal?
What is an Ordinal?

\[\text{obj} \rightarrow 1, \{\emptyset\} \rightarrow \emptyset, 0\]
What is an Ordinal?

- 2, \{\{∅\}, ∅\}
- 1, \{∅\}
- ∅, 0
What is an Ordinal?

- $\omega + 1$, $\{\{\omega\}, \omega\}$
- $\omega$
- $\vdots$
- $2$, $\{\{\emptyset\}, \emptyset\}$
- $1$, $\{\emptyset\}$
- $\emptyset$, $0$
What is an Ordinal?

An ordinal is a mathematical concept used in set theory. Ordinals are used to represent the order type of a well-ordered set. They are crucial in various areas of mathematics, including set theory and category theory. The set of all ordinals forms a proper class, which is not a set.

In this diagram, we have:
- The set of all ordinals, denoted as \( \{x | \text{ordinal } x \} \)
- The set containing the ordinal \( \omega + 1 \) and \( \{\omega, \omega\} \)
- The set containing the ordinal \( \omega \)
- The set containing the ordinal \( 2 \) and \( \{\emptyset, \emptyset\} \)
- The set containing the ordinal \( 1 \) and \( \emptyset \)
- The set containing the ordinal \( \emptyset \) and \( 0 \)
Operations On Ordinals

Successor of $\alpha$

$\{\alpha\} \cup \alpha$, i.e. $(\alpha; \alpha)$

- $\emptyset, 0$
- $1, \{\emptyset\}$
- $2, \{\{\emptyset\}, \emptyset\}$
- $\omega$
- $\omega + 1, \{\{\omega\}, \omega\}$

$\{x | (\text{ordinal } x)\}$
Operations On Ordinals

Successor of \( \alpha \)

\( \{\alpha\} \cup \alpha \), i.e. \((\alpha; \alpha)\)

[alpha:ordinal]

(ordsucc alpha):ordinal

\( \omega + 1, \{\{\omega\}, \omega\} \)

\( \omega \)

\( 2, \{\{\psi\}, \psi\} \)

\( 1, \{\emptyset\} \)

\( \emptyset, 0 \)
Operations On Ordinals

Successor of $\alpha$

\[ \{\alpha\} \cup \alpha, \text{i.e.} \ (\alpha; \alpha) \]

$\alpha$: ordinal

(ordsucc $\alpha$): ordinal

\(<(\alpha; \alpha), \text{ (ordsuccOrdinal} \ \alpha)\)>

($\alpha; \alpha$) corresponds to \(\text{(setadjoin } \pi_1(\alpha) \pi_1(\alpha))\)

\(\emptyset, 0\)
Operations On Ordinals

Successor of $\alpha$

$\{\alpha\} \cup \alpha$, i.e. $(\alpha; \alpha)$

$\langle \alpha; \alpha \rangle$, $\langle$ordsuccOrdinal $\alpha \rangle$

$\{x|($ordinal $x)$\}

$\omega + 1$, $\{\omega\}, \omega$

$\omega$

$\emptyset, 0$
Operations On Ordinals

Supremum of a set $X$ of ordinals: $\bigcup X$

- $\emptyset, 0$
- $1, \{\emptyset\}$
- $2, \{\{\emptyset\}, \emptyset\}$
- $\omega$

\{x|(ordinal x)\}
Operations On Ordinals

Supremum of a set $X$ of ordinals: $\bigcup X$

$(\text{ordsup } X): \text{ordinal}$
Operations On Ordinals

Supremum of a set $X$ of ordinals: $\bigcup X$

$\omega$

$[X: \{A \mid \forall x \in A. \ (\text{ordinal } x)\}]$

$(\text{ordsup } X): \text{ordinal}$

$\prec (\bigcup X), (\text{setunionOrdinal } X)\succ$

$1, \{\emptyset\}$

$\emptyset, 0$
Supremum of a set $X$ of ordinals: $\bigcup X$

\[
\{x \mid \text{ordinal } x\}
\]

(setunionOrdinal alpha)

(ordinal)

(ordsup X)

(setunionOrdinal X)

< (setunionOrdinal X)
To define successor on ordinals, we used a proof term.

This can be declared as a claim:

\[
[\alpha : \text{ordinal}] \\
(\text{ordsuccOrdinal } \alpha) : \vdash (\text{ordinal} \\
(\alpha;\alpha))?
\]

We then prove the claim interactively to obtain the proof term.
Making Claims

Actually, we split this into three lemmas.

(ordsuccOrdinalLem1 alpha) : \( \vdash \)
(\( \text{transitiveset (alpha ; alpha)} \))?

(ordsuccOrdinalLem2 alpha) : \( \vdash \)
(\( \text{stricttotalorderedByIn (alpha ; alpha)} \))?

(ordsuccOrdinalLem3 alpha) : \( \vdash \)
(\( \text{wellorderedByIn (alpha ; alpha)} \))?
Consider First Claim:

(ordsuccOrdinalLem1 alpha):\[\vdash \]
(transitiveset (alpha ; alpha))? 

"If $\alpha$ is an ordinal, then $\{\alpha\} \cup \alpha$ is a transitive set."
prove ordsuccOrdinalLem1

Given name for ordinal > alpha

> pplan

Support (Objects, Assumptions and Derived Facts in Context):

alpha : ordinal

fact0 : |- (ordinal alpha)

Goal (What you need to show): |- (transitiveset (alpha; alpha))

The goal has an abbreviation transitiveset at the head. Unfold it, working backwards.
Proving Claims Interactively

>foldhead
OK

>pplan

Support (Objects, Assumptions and Derived Facts in Context):

alpha:ordinal

fact0: |- (ordinal alpha)

Goal (What you need to show):  |- (dall (alpha;alpha) (\ x2.(x2<=(alpha;alpha))))

New Goal:  \( \forall x \in \{\alpha\} \cup \alpha . (x \subseteq (\{\alpha\} \cup \alpha)\).
New Goal: $\forall x \in \{ \alpha \} \cup \alpha . (x \subseteq (\{ \alpha \} \cup \alpha))$.
Introduce the quantifier.

> intro

OK

Give name for (in (alpha; alpha)) > x
Proving Claims Interactively

> pplan

Support ...:
alpha: ordinal
fact0: |- (ordinal alpha)
x: (in (alpha; alpha))  Assume \( x \in (\{\alpha\} \cup \alpha) \)
fact2: |- (x::(alpha;alpha))

Goal (What you need to show): |- (x<=(alpha;alpha))  Show \( x \subseteq (\{\alpha\} \cup \alpha) \)

Split into cases: \( x = \alpha \) or \( x \in \alpha \).
Proving Claims Interactively

> adjcases
OK
> pplan
Support ...:
alpha: ordinal
fact0: |- (ordinal alpha)
x: (in (alpha; alpha))
fact2: |- (x::(alpha; alpha))
ass0: |- (x==alpha)  

New Assumption  \( x = \alpha \).

Goal (What you need to show):  |-  
(x<=(alpha;alpha))
Proving Claims Interactively

> pstatus
0) (ass0 fact2 x fact0 alpha) |-
   (x<=(alpha;alpha))
1) (fact2 x fact0 alpha) x4:|-
   (x::alpha)|-
   (x<=(alpha;alpha))

There are now two gaps to fill corresponding to the two cases:

0) \( x = \alpha \)

1) \( x \in \alpha \)
Proving Claims Interactively

Current Case: $x = \alpha$

Current Goal: $x \subseteq (\{\alpha\} \cup \alpha)$

Idea: Use $\alpha \subseteq (\{\alpha\} \cup \alpha)$.

>`fact
Enter Proposition>` $(\alpha \leq (\alpha; \alpha))$

Correct.
The `fact` command tells Scunak to look for something to justify the given proposition in context.
Proving Claims Interactively

>pplan
Support ...:
alpha : ordinal
fact0 : |- (ordinal alpha)
x : (in (alpha;alpha))
fact2 : |- (x::(alpha;alpha))
ass0 : |- (x==alpha)
fact4 : |- (alpha<=(alpha;alpha))

Goal (What you need to show): |- (x<=(alpha;alpha))

Show: x ⊆ (α;α)

>d

Done with subgoal! Scunak finishes
Next Case: \( x \in \alpha \)

Goal: \( x \subseteq (\{\alpha\} \cup \alpha) \)

Idea: Use transitivity of ordinal \( \alpha \).

> fact

Enter Proposition> (x<=alpha)

Correct.

Scunak justifies \( x \subseteq \alpha \) using \( x \in \alpha \) and the fact that \( \alpha \) is an ordinal.
Proving Claims Interactively

>ppplan
Support ...:
alpha: ordinal
fact0: |- (ordinal alpha)
x: (in (alpha; alpha))
fact2: |- (x::(alpha; alpha))
ass1: |- (x::alpha)
fact5: |- (x<=alpha)

Goal (What you need to show): |- (x<=(alpha; alpha))

Show: $x \subseteq \alpha$

This gap is trivial enough...
Proving Claims Interactively

> d

Scunak Finishes

Done with subgoal!

Successful Term: Proof Term

\((\lambda \, x_0. \text{transitiveset}\#F \, (x_0; x_0) \, (\text{dallI} \, (x_0; x_0) \\
(\lambda x_1. \, (x_1<=(x_0; x_0)) \, (\lambda x_1. \text{setadjoinE} \, x_0 \\
x_0 \, x_1 \, x_1 \, (x_1<=(x_0; x_0)) \, (\lambda x_2. \text{equivEimp1} \\
(x_0<=(x_0; x_0)) \, (x_1<=(x_0; x_0)) \, (\text{subset}\#\text{Cong} \\
x_0 \, x_1 \, (\text{symeq} \, x_1 \, x_0 \, x_2) \, (x_0; x_0) \, (x_0; x_0) \\
(eqI \, (x_0; x_0)) \, (\text{setadjoinSub} \, x_0 \\
x_0)) \, (\lambda x_2. \text{setadjoinSub2} \, x_1 \, x_0 \, x_0 \\
(\text{ordinalTransSet1} \, x_0 \, x_1 \, x_2))))\)\)
Proving Claims Interactively

What facts were used in the proof?

transitiveset#F : Definition of transitiveset

dallI : Forall introduction

setadjoinE : Cases rule for $x \in (A; B)$

setadjoinSub : $B \subseteq (A; B)$.

setadjoinSub2 : If $Y \subseteq B$, then $Y \subseteq (A; B)$

ordinalTransSet1 : If $\beta$ is an ordinal and $y \in \beta$, then $y \subseteq \beta$.

+ some equality reasoning
Currently Proven

- $\emptyset$ is an ordinal.
- If $\alpha$ is an ordinal, then $\alpha \notin \alpha$.
- If $\alpha$ is an ordinal, then $(\alpha; \alpha)$ is an ordinal.
- If $\alpha$ is an ordinal and $x \in \alpha$, then $x$ is an ordinal.
- Every ordinal is a successor or a limit ordinal.
- If $\alpha$ is an ordinal and $\alpha \in x$, then $x \notin \alpha$.
- If $\alpha$ and $\beta$ are ordinals and $\alpha$ is a proper subset of $\beta$, then $\alpha \in \beta$.
- If $\alpha$ and $\beta$ are ordinals, then $\alpha \cap \beta$ is an ordinal.
- If $\alpha$ and $\beta$ are ordinals, then $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$. 
Currently Proven

- Any set of ordinals is well-ordered by \( \in \).
- No ordinal is between \( \alpha \) and the successor of \( \alpha \).
- If \( X \) is a set of ordinals, then \( \bigcup X \) is an ordinal.
- **Ordinal Induction:** Let \( \phi \) be a property. Assume for every ordinal \( \alpha \), \( \phi(\alpha) \) holds if \( \phi(x) \) holds for every \( x \in \alpha \). Then \( \phi(\alpha) \) holds for all ordinals \( \alpha \).
Conclusion

Scunak can encode ordinals nicely because:

- The untyped set theory allows the “ill-typed” definition of the ordinals.
- The dependent type theory allows the class of ordinals to be used as a type.

Also,

- The interactive prover allows reuse of previously proven facts without explicit mention of the names of the facts.

Future Work:

- Counting as high as I can.