Semantics of Higher-Order Logic
Exercise Sheet 1

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In class we proved the Surjective Cantor Theorem: There is no surjection from $A$ onto $\mathcal{P}(A)$.

Here we consider the following False Generalization of Cantor’s Theorem:
For any set $A$ and subset $S$ of $\mathcal{P}(A)$, there is no surjection from $A$ onto $S$.

Exercise 1 (5 Points) Give a counterexample to the false generalization of Cantor’s Theorem. That is, give a set $A$, subset $S$ of $\mathcal{P}(A)$, and a surjection from $A$ onto $S$.

Exercise 2 (5 Points) The following is not a proof of the false generalization of Cantor’s Theorem. Indicate which step is incorrect and explain why the step is incorrect.

1. Assume $g : A \rightarrow S$ is surjective.
2. Let $D$ be \{ $x \in A$ | $x \notin g(x)$ \}.
3. Since $g$ is surjective, there is some $X \in A$ such that $g(X) = D$.
4. $X \in D$ iff $X \in g(X)$ since $g(X) = D$.
5. $X \in D$ iff $X \notin g(X)$ by the definition of $D$.
6. Hence $X \in g(X)$ iff $X \notin g(X)$, which is a contradiction.

Exercise 3 (10 Points) The Injective Cantor Theorem states that there is no injection from $\mathcal{P}(A)$ into $A$. Complete the following proof of the Injective Cantor Theorem by filling in the missing steps. Indicate explicitly any place where injectivity of $h$ is used.

1. Assume $h : \mathcal{P}(A) \rightarrow A$ is injective.
2. Let $D$ be the set \{ $h(X) \in A$ | $X \in \mathcal{P}(A)$, $h(X) \notin X$ \}.
3. Either $h(D) \in D$ or $h(D) \notin D$.
4. Case 1: Assume $h(D) \in D$.
5. ... .
6. Case 2: Assume $h(D) \notin D$.
7. ... .

For any function $f : A \rightarrow B$ and element $b \in B$, $f^{-1}(b)$ denotes the set \{ $a \in A$ | $f(a) = b$ \}.

For any set $A$ and subset $X \subseteq A$, we use $\chi_X : A \rightarrow \{ T, F \}$ to denote the characteristic function defined by

$$
\chi_X(a) = \begin{cases} 
T & \text{if } a \in X \\
F & \text{if } a \notin X 
\end{cases}
$$

There are two important mathematical principles you should make sure you always know. They are both used in the exercise below.

Set Extensionality: Two sets are equal if they contain the same elements. That is, to show two sets $X$ and $Y$ are equal, one can show $z \in X$ iff $z \in Y$ for all $z$. Equivalently, to show two sets $X$ and $Y$ are equal, one can show $X \subseteq Y$ and $Y \subseteq X$.

Functional Extensionality: Two functions are equal if they both output the same value given the same input. That is, to show two functions $f, g : A \rightarrow B$ are equal, it is enough to show $f(a) = g(a)$ for all $a \in A$.

Exercise 4 (30 Points) Suppose $\mathcal{D}$ is the standard frame with $D_o = \{ T, F \}$ and $D_i = \mathbb{IN} = \{0, 1, 2, \ldots \}$.

1. (5 Points) Prove the following:
(a) For all \( f \in \mathcal{D}_{i \rightarrow o} \), \( f^{-1}(T) \in \mathcal{P}(\mathbb{N}) \).
(b) For all \( f \in \mathcal{D}_{i \rightarrow o} \), \( f = \chi_{f^{-1}(T)} \).

2. **(5 Points) Prove**
(a) For all \( X \in \mathcal{P}(\mathbb{N}) \), \( \chi_X \in \mathcal{D}_{i \rightarrow o} \).
(b) For all \( X \in \mathcal{P}(\mathbb{N}) \), \( X = (\chi_X)^{-1}(T) \).

3. **(15 Points)** A binary relation \( R \) on \( \mathbb{N} \) is a set of pairs \( \langle n, m \rangle \) of natural numbers. Let \( \text{Relns} \) be the set of binary relations on \( \mathbb{N} \). That is, \( \text{Relns} := \mathcal{P}(\mathbb{N} \times \mathbb{N}) \) (the set of all sets of pairs of natural numbers).

Note that each \( g \in \mathcal{D}_{i \rightarrow o} \) is a function from \( \mathbb{N} \) to \( \mathcal{D}_{o} \). For each \( n \in \mathbb{N} \), \( g(n) \) is also a function, this time from \( \mathbb{N} \) to \( \mathcal{D}_o \). Hence for each \( n, m \in \mathbb{N} \), \( g(n)(m) \) – or we could write \( (g(n))(m) \) – is a member of \( \mathcal{D}_o \).

Construct functions \( \varphi : \mathcal{D}_{i \rightarrow i \rightarrow o} \rightarrow \text{Relns} \) and \( \psi : \text{Relns} \rightarrow \mathcal{D}_{i \rightarrow i \rightarrow o} \) such that
(a) For every \( g \in \mathcal{D}_{i \rightarrow i \rightarrow o} \), \( \psi(\varphi(g)) = g \) and for all \( n, m \in \mathbb{N} \), \( g(n)(m) = T \) iff \( \langle n, m \rangle \in \varphi(g) \).
(b) For every \( R \in \text{Relns} \), \( \varphi(\psi(R)) = R \) and for all \( n, m \in \mathbb{N} \), \( \psi(R)(n)(m) = T \) iff \( \langle n, m \rangle \in R \).

4. **(5 Points)** Explain how the set \( \mathcal{D}_{i \rightarrow i \rightarrow o} \) corresponds both to the set of all binary relations on \( \mathbb{N} \) and to the set of all functions from \( \mathbb{N} \) to \( \mathcal{P}(\mathbb{N}) \).